

Consensus formation on simplicial complex of opinions

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Abstract

Geometric realization of opinion is considered as a simplex and the opinion space of a group of individuals is a simplicial complex whose topological features are monitored in the process of opinion formation. The agents are physically located on the nodes of the scale-free network. Social interactions include all concepts of social dynamics present in the mainstream models augmented by four additional interaction mechanisms which depend on the local properties of opinions and their overlapping properties. The results pertaining to the formation of consensus are of particular interest. An analogy with quantum mechanical pure states is established through the application of the high dimensional combinatorial Laplacian.

1 Introduction

The interest of physics community in modeling problems of social dynamics, in particular the application of statistical physics concepts and methods on the modelling of opinion formation and dynamics gave birth to many simple opinion models [1]. We mention here just those which had major influence on advances in this field such as the Voter model [2], Galam model [3], social impact model [4], Sznaid model [5], Deffuant model [6], and the Krause-Hegselman model [7]. Some of these models consider population of individuals (agents) with discrete opinions represented as integers, for example (+1/-1) [2], [3], [4], [5], while others consider population of individuals with continuous, bounded range of opinions [6], [7]. In the so called consensus models [1] computer simulation of opinion dynamics starts with Monte Carlo simulation of randomly distributed opinions over the population of agents located on the nodes of a graph. The graph types include small world networks [8], scale-free networks [9], and fully connected graphs enabling each agent to interact with every other agent [6]. Interaction between agents differs from model to model and at the end of simulation leads to the state which may be characterized as consensus (single opinion state), polarization (two opinions), or anarchy (diversity of opinions). The majority of models are characterized by interactions between agents which may include social phenomena characterizing realistic

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communication situations such as transmission of information from the individual to his(her) neighbors, social influence, homophily and bounded confidence [1].

The social actions of an individual reflect his (her) opinions as systems of beliefs on different issues (or generally subjects in their broadest meaning) which may or may not have an empirical background. In this sense the opinions involve an individual's perception and knowledge or emotional conditions about the likelihood of events or relationships regarding some topic, and they also may involve evaluations of an event or an object [10]. An individual may have an opinion about different issues, such as religious matters, treatment of criminals or expansion of crime, political issues including political parties, advertising, movies, ... or nearly anything. In order to formulate or express an opinion on certain topic an individual evaluates a set of interconnected judgements. When, for example, a person talks, writes a mail or an article about his(her) opinion, he(he) actually sends a "package" of interconnected judgements which form an opinion. In this sense an opinion is not a collection of separate but interrelated and interconnected judgements which form an authentic opinion. Let us illustrate this with an example considering the issue of crime expansion. An individual forms an opinion on the crime expansion by forming and expressing judgements on topics of *organized crime*, *corruption* and *gambling*, to name only a few causes responsible for the increase of crime rate. When the conversation topic is the crime rate a person expresses his opinion using all three judgments in an interconnected manner and juxtaposes them under a certain relationship. Adding a new judgement on a possible crime cause, like *low incomes*, a person's opinion shifts to the new opinion since now four judgements are interconnected and interrelated. Of course, all four mentioned judgements can be treated as opinion issues on their own, however we will not discuss this case here since they are used here only for illustration purposes.

Collection of opinions of a large number of individuals represents an entanglement of overlapping opinions and shared judgements, the analysis of which requires a suitable mathematical framework which captures the essence of opinions and their formation. Let us call the set of all different opinions the opinion set and the set of all judgements the judgement set. One approach would consider these as two defining sets of bipartite graph. In this case the judgements are represented as vertices, and two vertices are connected by a bond if two corresponding judgements are part of the same opinion. Analyzing this structure from the graph-theoretic point of view the actual information about opinions is lost since only connectivities between judgements are considered. We can keep track of judgments as the basic constituent elements of opinions and their mutual relationships however the information about each opinion as a whole is unavailable. An alternative and more comprehensive way is to consider opinion and judgement sets from the point of view of combinatorial algebraic topology. In this approach the judgements are again represented as vertices, opinions are now represented as simplices [11], while the individuals remain located on the nodes of the scale-free network. A n -simplex represents an n -dimensional polytope which actually is the convex hull of its $n + 1$ vertices. Hence, a 2-simplex is a triangle, a 3-simplex is a tetrahedron and so on. A simplex is not only a graph but also includes higher dimensional faces and a space enclosed by the faces. Since simplices overlap by sharing vertices the collection of simplices together with their overlaps is called simplicial complex. The graph itself is a 1-dimensional simplicial complex. The simplicial complex has a three-fold mathematical property since it can be considered from a topological, combinatorial and algebraic aspect

and may be most efficiently analyzed using concepts combining these three mathematical disciplines. We consider the geometric realization of an opinion as a simplex and the opinion set as a simplicial complex formed through the overlapping connectivity structure of opinions which is realized through shared judgements. The simplicial complex of opinions also encloses an opinion space defined by the complex itself. The number of judgements that characterizes a single opinion need not be the same and it may vary from person to person. An example of two persons having opinions on the expansion of crime, represented geometrically as simplices, is presented in Figure 1. The case when they share certain judgements is presented on the left and the case when they do not share any judgements is presented on the right.

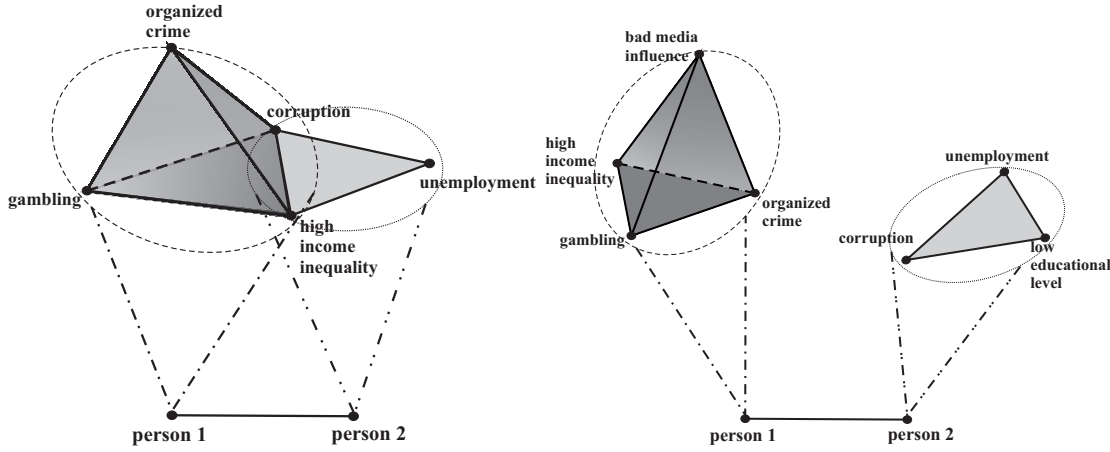


Figure 1 Two persons expressing opinion on the expansion of crime. The case of overlapping judgements is on the left and the case of non overlapping on the right.

Communication between individuals alters the shape of the opinion space and by numerically evaluating certain topological quantities and topological invariants [11] these changes can be monitored and predictions about the final state can be made, for example, whether the consensus will be achieved or not. The predictions are further supported by the existence of pure opinion states defined through the properties of properly normalized Combinatorial Laplacian as explained further in the text.

Social entities in the form of individuals (or agents) are the carriers of social interpersonal interaction [14] so that the opinion of a single person or a group of persons may be under the influence of another individual or another group of individuals. Interactions may be of dyadic type (between two persons), triadic type (between three persons) or in general of n -adic type which includes n persons and clearly social interaction depends on the interpersonal connectivity of persons. In the model presented here social interactions include all concepts of social dynamics present in the mainstream models such as information flows directed outwards (i.e. from the agent to its neighbors), social influence, homophily and bounded confidence. There are four additional interaction and communication activities which may alter individual's opinion in this model. The result of the interaction to a large extent depends on whether two agents have the same or diverging opinions, or if they have different but to a certain extent overlapping opinions. If two agents have the same opinion they convince all their common neighbors to adopt this opinion. This type of interaction is in common with the interaction of the Sznajd model [5]. If two agents have different but overlapping

opinions, the result of their interaction depends on the extent of the overlap, and can be twofold. The agents can either exchange one of the judgements forming compromise or they can unite their opinions by uniting judgements (forming joint opinion) and the probability of the two outcomes is proportional to the extent of the overlap. In both cases new opinions are formed and added to the opinion space. This type of interaction is similar to the Axelrod model [15] with some important differences. In Axelrod model the culture is defined as a vector $F = (f_1, f_2, \dots, f_F)$ and each entry f_i (called cultural feature) can take a value from the so called trait set $f_i \in \{\tau_1, \tau_2, \dots, \tau_T\}$, whereas in our model the opinion represented as simplex, is an unordered set of judgements. Furthermore, the length of vector F in the Axelrod model is fixed while in our model the dimension of the opinions (number of judgements which characterize them) may vary. With respect to this property our model is somewhat closer to the naming game model [16], [17], although the tendency in this model is to decrease the inventory of word-associations to common words, while in our case the tendency is to increase the number of judgements which characterize the opinion. The opinions in our model may experience alteration in one additional way. The model allows for a new previously absent judgement to be added to the judgements set under specific conditions. Consequently, the agent's corresponding opinion changes and the new opinion is added into the complete set of opinions. This type of interaction mechanism has remote resemblance to the "cultural drift" mechanism introduced as a variant of the Axelrod model in [18], however the possibility to change the trait spontaneously is driven by random noise and in our model the change is caused by the local property of agent's opinion.

One of the aims of this study is also to promote the model founded on the structure of simplicial complex as the natural setting for opinion formation and other social network studies which offers a number of new aspects and results in comparison with the standard graph structure approach. The organization of the paper is as follows. Following a brief introduction to topological and geometrical features of simplicial complexes in Section 2, we present the model of opinion formation on simplicial complexes and the algorithm in a detailed manner in Section 3. The results of the simulation are presented in Section 4 and discussion of results and conclusion is presented in Section 5.

2 Simplicial complex

2.1 Definition and topological features

A short introduction to the topology of simplicial complexes presented in this Section covers basic properties and fundamental topological measures used in the analysis. More detailed description of terms and concepts used may be found in [11], [12] and [13].

Simplicial complexes are formed by simplices which may have different dimensions, and hence can be analyzed as multidimensional and multilevel objects. Given a set of points whose elements we call *vertices* $V = \{v_0, v_1, \dots, v_n\}$, any subset of $q + 1$ elements of this set $\{v_{\alpha_0}, v_{\alpha_1}, \dots, v_{\alpha_q}\}$ is called *q-dimensional simplex*, or simply *q-simplex*. A *p-simplex* σ_p is a *p-face* of a *q-simplex* σ_q , denoted by $\sigma_p \leq \sigma_q$, if every vertex of σ_p is also a vertex of σ_q , and if two simplices σ_q and σ_r have $p + 1$ common vertices then they share a *p-face*. A simplicial complex represents a collection of simplices together with their faces. In more formal terms a simplicial complex K on a finite set $V = \{v_1, \dots, v_n\}$ of vertices is a nonempty subset of

the power set of V , so that K is closed under the formation of subsets [19]. The maximal dimension of a simplex in K determines the dimension of the whole simplicial complex. A graph is a 1-dimensional simplicial complex with the same set of vertices and with simplices represented as edges of a graph. An illustration of the simplicial complex is presented in Figure 2.

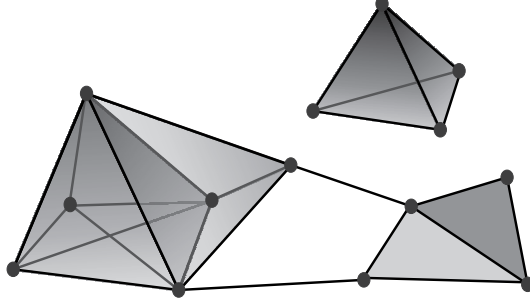


Figure 2 An illustration of simplicial complex formed by one 4-simplex, two 3-simplices, two 2-simplices, and two 1-simplices.

Throughout this presentation we adopt the notation where the subscript marks the dimension of a simplex and the superscript marks the numerical name of a simplex, hence the notation σ_q^i means the " q -dimensional simplex i ". Two simplices σ and ρ are q -connected [20] if there is a sequence of simplices $\sigma, \sigma^1, \sigma^2, \dots, \sigma^n, \rho$, such that any two consecutive ones share at least a q -face. The q -connectivity between simplices induces an equivalence relation on simplices of a complex K , since it is reflexive, symmetric, and transitive. This equivalence relation be denoted by γ_q so that

$$(\sigma^i, \sigma^j) \in \gamma_q \quad \text{if and only if } \sigma^i \text{ is } q\text{-connected to } \sigma^j.$$

Let K_q be the set of simplices in K with dimension greater than or equal to q . Then γ_q partitions K_q into equivalence classes of q -connected simplices. These equivalence classes are members of the quotient set K_q/γ_q and they are called the q -connected components of K . Every simplex in a q -component is q -connected to every other simplex in that component, but no simplex in one q -component is q -connected to any simplex on a distinct q -connected component. Changing the roles of simplices and vertices of simplicial complex K new simplicial complex K^{-1} is formed in which simplices are the vertices of K and vertices are the simplices of K . This new simplicial complex K^{-1} is called *the conjugate complex* of simplicial complex K [20]. Illustration of the original and the conjugate complex is presented in Figure 3 (a) and (b), respectively. Vertices are labeled by numbers (Figure 3(a)) and letters (Figure 3(b)), whereas simplices by $\sigma(i)$, where $i = a, b, c, \dots$ (Figure 3(a)) and $i = 1, 2, 3, \dots$ (Figure 3(b)).

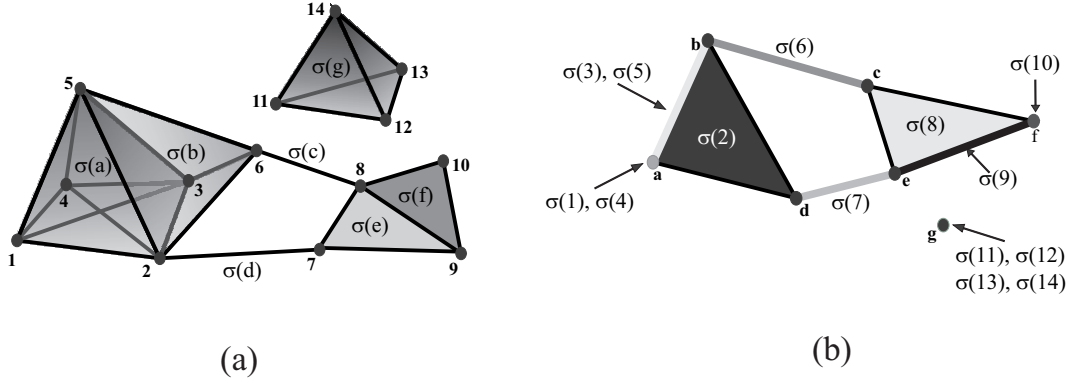


Figure 3 An illustration of simplicial complex (a) and its conjugate (b).

2.2 Structure properties of simplicial complexes

Q-vector (first structure vector). The cardinality of K_q/γ_q is denoted Q_q and is the number of distinct q -connected components in K . The value Q_q is the q^{th} entry of the so called *Q-vector (first structure vector)* [20], an integer vector with the length $\dim(K) + 1$. An example illustrating the partitioning the simplicial complex into q -connectivity classes and Q-vector is presented in Figure 4. Hence, Q-vector describes the structure of simplicial complex on different levels of connectivity. The notation is [20]:

$$\mathbf{Q} = (Q_{(\dim(K))}, Q_{(\dim(K)-1)}, \dots, Q_0) . \quad (1)$$

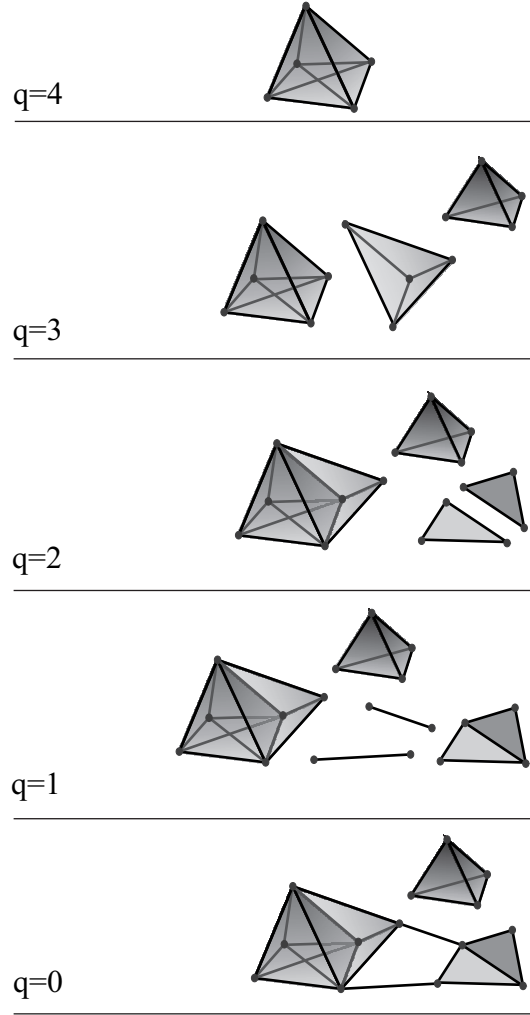


Figure 4 Graphical illustration of the Q-vector of simplicial complex from Figure 2.

Eccentricity. Define \hat{q} (top q) as the dimension of the simplex and define \check{q} (bottom q) as the largest dimension of faces which simplex σ share with other simplices, i.e. the largest q -nearness value. \check{q} is equivalent to the value of the q -level on which the simplex firstly connects to some other simplex. Then the eccentricity of simplex σ is defined as [11]:

$$ecc(\sigma) = \frac{\hat{q} - \check{q}}{\hat{q} + 1}. \quad (2)$$

$ecc(\sigma)$ measures the individuality of a simplex, and indicates degree of integrity of the simplex σ in a simplicial complex. The simplex which has $ecc = 0$ is completely integrated into the structure, i.e. the simplex is face of another simplex. The simplex which has $ecc = 1$ does not share vertices (faces) with any other simplex, i.e. it is completely disintegrated.

Vertex significance of a simplex. One vertex can be part of many simplices and a vertex weight θ provides information on the number of simplices which are created by that vertex. The sum of weights of the vertices which create simplex $\sigma_q(i)$ yields $\Delta(\sigma_q(i))$, and the vertex significance of the simplex is defined as [11]:

$$vs(\sigma_q(i)) = \frac{\Delta(\sigma_q(i))}{\max_k \Delta(\sigma_q(k))}, \quad (3)$$

where $\max_k \Delta(\sigma_q(k))$ is the maximal value of all $\sigma_q(i)$. The larger value of vs indicate larger importance of the simplex with respect to the vertices which create it in the sense that, compared to other simplices, it contains vertices which take part in the construction of the larger number of other simplices.

Combinatorial Laplacian. The matrix representation of the q^{th} Laplacian matrix of simplicial complex K is

$$\mathcal{L}_q = B_{q+1}B_{q+1}^T + B_q^TB_q.$$

The boundary operator which maps simplices of dimension $q + 1$ to simplices of dimension q is represented by the matrix B_q so that the rows of B_q are associated with simplices of dimension $q+1$ and columns are associated with simplices of dimension q . Graph represents a 1-dimensional simplicial complex since links (1-dim simplices) connect nodes (0-dimensional simplices) and the largest dimension of a simplex in the complex is 1. In B_1 , the matrix representation of the boundary operator ∂_1 , the rows are associated with edges and columns are associated with vertices so that the matrix B_1 is equal to the incidence matrix of an oriented graph. Therefore matrix representation of the combinatorial Laplacian is $\mathcal{L}_0 = B_1B_1^T$ and the matrix elements are

$$(\mathcal{L}_0)_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j \\ -1, & \text{if } v_i \sim v_j \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

where $\deg(v_i)$ is vertex degree (that is number of neighbors of a vertex v_i) and the relation $v_i \sim v_j$ is the adjacency relation between vertices v_i and v_j . Clearly, the entries of the 0-dimensional combinatorial Laplacian are the same as the graph Laplacian entries defined in the usual way via expression $L_{graph} = D - A$, where diagonal entries of matrix D are equal to the vertex degrees ($D_{ii} = \deg(v_i)$) and nondiagonal entries are zeros, and the entries of matrix A are $(A)_{ij} = 1$ if $v_i \sim v_j$, $(A)_{ij} = 0$ if vertices v_i and v_j are not neighbors, and $(A)_{ii} = 0$ (undirected, unweighted, without loops and multiple edges graph) [12].

For the general case let us assume that K is an oriented simplicial complex, q is an integer with $0 < q \leq \dim(K)$, and let $\{\sigma^1, \sigma^2, \dots, \sigma^n\}$ denote the q -simplices of complex K , then

$$(\mathcal{L}_q)_{ij} = \begin{cases} \deg_U(\sigma^i) + q + 1, & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \sigma^i \text{ and } \sigma^j \text{ are} \\ & \text{not upper adjacent but} \\ & \text{have a similar common} \\ & \text{lower simplex} \\ -1, & \text{if } i \neq j \text{ and } \sigma^i \text{ and } \sigma^j \text{ are} \\ & \text{not upper adjacent but} \\ & \text{have a dissimilar common} \\ & \text{lower simplex} \\ 0, & \text{if } i \neq j \text{ and } \sigma^i \text{ and } \sigma^j \text{ are} \\ & \text{upper adjacent or are not} \\ & \text{lower adjacent} \end{cases}.$$

3 The Model

Opinions are modeled as simplices and social interaction mechanisms were presented in the Introduction. For simplicity, we have chosen that initially each opinion is characterized by the same number of judgements implying that each simplex has the same dimension q_{in} at the beginning of the simulation, although the case when each simplex has different dimension can be easily implemented. The dimension q_{in} as well as the total number of judgements (i.e. vertices) and number of opinions associated with agents will be changed due to interaction mechanisms. The relationship between simplices (opinions) and vertices (judgements) is captured in the so called incidence matrix Λ [11], [20], in which rows are associated to simplices and columns to vertices, such that the matrix element is $\Lambda_{ij} = 1$ if vertex j is part of the simplex i , and 0 otherwise. From this matrix all other quantities describing the structure of the interaction may be derived. During the simulation the size of incidence matrix changes since vertices (judgements) can appear or disappear and simplices (opinions) may change and also emerge or dissolve. The actual stages of the algorithm are presented in the chronological order:

- (1) The agents are located on the sites of the scale-free complex network (graph).
- (2) Fix two parameters, initial dimension of simplices q_{in} and initial number of judgements v_{in} ($q_{in} \leq v_{in}$).
- (3) To each agent associate $q_{in} + 1$ different random numbers between 1 and v_{in} . Each different unordered set of $q_{in} + 1$ judgements defines a single simplex, and associated to each agent is an opinion defined by the same number of judgements.
- (4) Form initial simplicial complex of opinions O_{in} (the initial incidence matrix) from different unordered sets generated at step (3). Calculate initial Q -vector, \mathbf{Q}_{in} . If all entries of \mathbf{Q}_{in} are equal to 1 simulation stops, otherwise continue to step (5).
- (5) Randomly choose one of the agents i . If the simplex of agent's opinion $\sigma_q(o_i^1) = \langle v_0, v_1, v_2, \dots, v_q \rangle$ has the associated vertex significance equal to 1, i.e. $vs(\sigma_q(o_i^1)) = 1$, then extend his (her) opinion by an additional judgement not previously present in the initial set of judgements so that a new opinion $\sigma_{q+1}(o_i^2) = \langle v_0, v_1, v_2, \dots, v_q, v_{q+1} \rangle$ is formed. As a result of this process sets of opinions and judgements are enlarged by adding new simplex and new judgement and the dimension of simplicial complex of opinions is increased by 1. If $vs(\sigma_q(o_i^1)) < 1$ nothing happens and the algorithm advances to step (6).
- (6) Randomly choose one of the agents i , and one of its nearest neighbors j . The opinion simplices of these two agents are $\sigma_{q_1}(o_i^1)$ and $\sigma_{q_2}(o_j^2)$. We distinguish between two cases:
 - (a) If opinions (i.e. simplices) of agents i and j are the same ($\sigma_{q_1}(o_i^1) = \sigma_{q_2}(o_j^2)$) the agents convince all their common nearest neighbors to adopt this opinion. The result of this step is the change of opinions of the neighboring agents i and j without any alteration of the simplicial complex of opinions.
 - (b) If opinions (i.e. simplices) of agents i and j are not the same but they overlap the result of their communication can be either compromise or joint opinion. If the overlap assumes sharing $f_{ij} + 1$ judgements, i.e. simplices corresponding to opinions share f_{ij} -face, determine the *degree of compromise* for each of the opinions. First define the *overlap degrees* from the aspect of opinions of agents i and j as $\omega_i = f_{ij}/q_i$ and $\omega_j = f_{ij}/q_j$ respectively, where q_i and q_j are dimensions of opinion simplices of agents i and j , respectively. Then the

degree of compromise is defined as:

$$\phi = \frac{(\omega_i + \omega_j)}{2}. \quad (5)$$

Generate random number r between 0 and 1 from the uniform distribution. If $r \leq \phi$ then the agents merge their opinions into a new opinion of dimension equal to $q_i + q_j - f_{ij}$ ($> q_i, q_j$) formed from the union of individual judgements. Associate this new opinion with each of the agents (i.e. the agents embrace this new opinion) and add this new simplex to the simplicial complex of opinions. If $r > \phi$ then the opinion of one of the agents is changed by removing a judgement (not from the overlap set) and then adopting a randomly chosen judgement (not from the overlap set) from the opinion of other agent. The results of this process is the change of the opinion of one agent and the addition of the new simplex into simplicial complex of opinions. The stochastic criterion used at this stage favors the joint opinion.

At each simulation stage the Q -vector is evaluated and simulation stops when all entries of Q -vector are equal to 1. If this condition is fulfilled the consensus will be achieved after long enough time. The value equal to 1 for all Q -vector elements implies that there is only one connectivity class at each connectivity level. When new opinions are formed they will appear in some of the already present connectivity class and the number of opinions in connectivity classes decrease.

Since the opinion space is geometrically represented as simplicial complex we define the pure geometrical states of the opinion space in the analogy with the quantum mechanical pure states. Taking a q -Laplacian matrix as a density matrix at dimension q , a simplicial complex of opinions is formed by pure states if the following relation is satisfied:

$$Tr(L_q^n) = Tr([L_q^n]^2),$$

where $Tr(\quad)$ is trace of the matrix, and $L_q^n (= A \cdot L_q)$ is the properly normalized Laplacian matrix for dimension q . Choosing the normalization constant of the Laplacian matrix as

$$A = \frac{\sum_i n_i d_i}{\sum_i n_i d_i^2},$$

where d_i are diagonal elements and n_i is multiplicity of the d_i -th diagonal element, the "trace" condition is satisfied. In this case we say that simplicial complex is in the pure state if it is formed by the collection of disconnected simplices. At the end of simulation we check whether the opinion space consists of pure states or not.

4 Results of simulation

In order to examine the consequences of the interaction mechanisms on changes of the opinion space, we have performed a simulation with S agents located at the sites of the scale-free network with well defined communities, called modules [21]. In the construction of clustered modular scale-free network we used the algorithm introduced in [22] with the following model parameters: M - mean number of links per node, P_0 - probability of module formation, α - rewiring parameter which provides higher (lower) clustering coefficient and N the number

of nodes. Numerical values of parameters are $M = 5$, $P_0 = 0.007$, $\alpha = 0.6$, and $N = 1000$. This network has high clustering coefficient and 7 modules. In the opinion space initial dimension of each simplex (opinion) is $q_{in} = 4$ and we varied the initial number of vertices (judgements) from the set $v_{in} = [4, 5, 6, 7, 8]$. The focus of our interest lies in monitoring changes in the opinion space regarding the number of opinions associated to agents S (some opinions disappear and some new opinions are added), the number of judgements v (new judgements are added), the opinion o_m associated to the maximal number of agents n_m , and topological and geometric properties of simplex associated with the opinion o_m such as eccentricity and vertex significance of simplex. The maximal number of different realized opinions associated to agents during the simulation is denoted by S_{max} , and the number of different opinions associated to agents at the end of simulation is denoted by S_{end} . We recall that the simulation stops when consensus state is reached which in the topology of the opinion space implies that the simplicial complex of opinions is geometrically connected at all levels of connectivity (recall the structure of the Q -vector). We have tested this assumption for all values from the set v_{in} and in each case the consensus was reached.

Since the initial number of judgements v_{in} was changed in simulations the dependence of S_{max} and S_{end} on the ratio q_{in}/v_{in} was tested and is presented at Fig. 5 left and Fig. 5 right, respectively. Both $S_{max}(\frac{q_{in}}{v_{in}})$ and $S_{end}(\frac{q_{in}}{v_{in}})$ relationships are in good agreement with the power law fit with exponents 6.24 and 5.86, respectively. Their ratio $5.86/6.24 = 0.93$ is equal to the exponent of the power law dependence $S_{end}(S_{max})$ which means that very small portion of opinions disappeared, primarily by the convincing process. After reaching maximum the convincing process assumes more important role so that some opinions disappear and connectivity of opinion space at all levels of connectivity decreases.

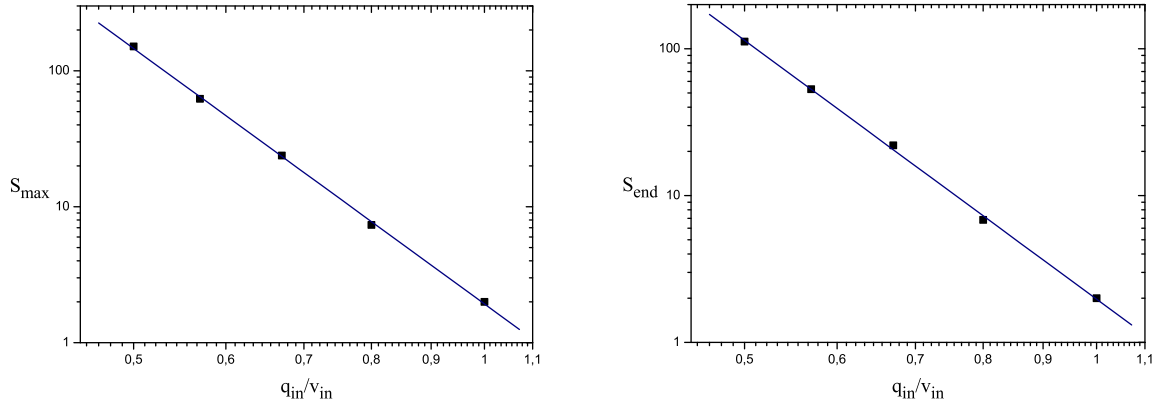


Figure 5 Maximal number of different realized opinions associated to agents during the simulation (S_{max}) (left), and the number of different opinions associated to agents at the end of simulation (S_{end}) (right) as a function of the parameter ratio q_{in}/v_{in} .

The relationship between initial number of vertices (judgements) v_{in} and the number of vertices (judgements) at the end of the simulation v_{end} obeys a power law with exponent 1.1 shown in Fig. 6. This value of exponent suggests small influence of newly formed judgements on topology of the opinion space.

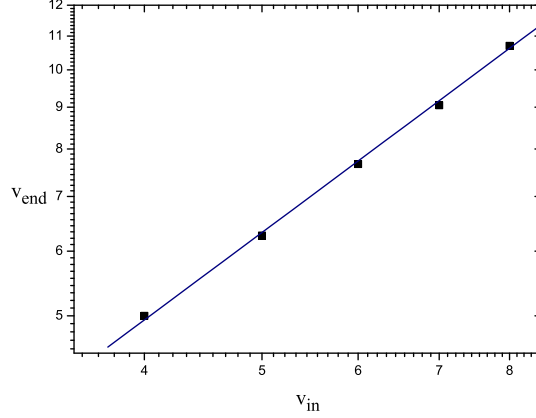


Figure 6 Relationship between initial number of vertices (judgements) v_{in} and the number of vertices (judgements) at the end of the simulation v_{end} .

Fig. 7 displays dependence of the maximal number of agents n_m associated with the single opinion on the initial parameter q_{in}/v_{in} . The relationship displays the power law with the exponent -0.82 . It turns out that keeping q_{in} constant and increasing v_{in} and hence the number of different opinions, the largest number of agents associated to the single opinion n_m grows. This is somewhat paradoxical since it implies that in spite of the growing number of opinions and hence choices for the agents the number of individuals adhering to and/or adopting the same opinion increases.

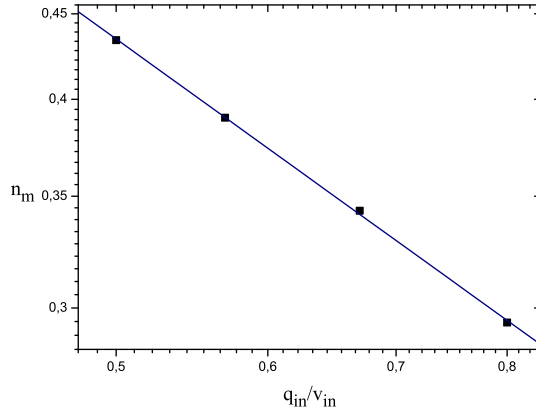


Figure 7 Dependence of the maximal number of agents n_m associated with the single opinion on the initial parameter ratio q_{in}/v_{in} .

At the end of the simulation of special interest is the eccentricity and vertex significance of the opinion o_m adopted by the largest number of agents. The eccentricity value for all v_{in} is around 0, which means that this opinion is well integrated into the opinion structure, and that it shares almost all judgements with the other opinions. The vertex significance value for all v_{in} is in the range $8.1 \leq vs \leq 9.1$ confirming the importance of the opinion o_m with the largest number of individuals who embrace it. The fact that it shares all of its

judgements with other opinions but still attracts the largest number of individuals bolsters its importance. The opinion space in all considered cases consists of pure states at the end of the simulation.

5 Conclusion

An opinion dynamics model which assumes opinions as unordered sets of judgements is developed. The opinion space is mapped to simplicial complex so that the analysis uses concepts of combinatorial algebraic topology, primarily relying on the Q-vector. Through the interaction and communication mechanisms opinions appear and disappear causing changes in the topology of opinion space monitored by evolution of the Q-vector. When all entries of Q-vector are equal to 1 the consensus is achieved.

We have calculated the maximal number of realized opinions associated to agents during the simulation, the number of realized different opinions associated to agents at the end of simulation (when entries of Q-vector are equal to 1), number of judgements at the end of the simulation, and the largest number of agents associated to the single opinion. All these quantities display power law dependence as functions of v_{in} where v_{in} is the initial number of vertices. Particularity interesting is the increase of the largest number of agents associated to the same single opinion as a function of increasing v_{in} . It turns out that the larger the initial difference between opinions the larger number of agents are associated with the same opinion, which is in contradiction with the results for the Axelrod model [15]. These results are valid for only one fixed value of q_{in} and five small values of v_{in} and the future work will be include the change of these two initial values and examination whether the above quantities follow the same behavior. The influence of external factors such as the mass-media or marketing may be easily introduced in the model by representing them as an external simplicial complex. The opinion associated with the largest number of agents adhering to the same opinion is well integrated into the structure of opinions and it would be interesting to check whether the consensus is achieved over this opinion. The results presented here indicate that at the end of simulation the opinion space is formed by the pure states and that the future opinion dynamics is strongly dominated by the convincing process. The intention of the authors was not to predict any real situation with this model but to highlight coupling of different communication mechanisms and the structure and evolution of the opinion space through the applications of the simplicial complex.

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